COHOMOLOGY OF NONRATIONAL TORIC MANIFOLDS.

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A *complex manifold with maximal torus action* M is given via the following data:

- G is a compact torus;
- Δ is a nonsingular fan in Lie algebra $\mathfrak{g}$ of G with respect to the lattice $\ker \exp_G$;
- $\mathfrak{h} \subset \mathfrak{g}^{\mathbb{C}}$ is a complex subspace such that the restriction to $\mathfrak{h}$ of real projection $p\colon \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}$ is injective;
- $\Sigma := q(\Delta) = \{q(\sigma) \subset \mathfrak{g}/p(\mathfrak{h})\colon \sigma \in \Delta\}$ is a complete fan, and the map $\Delta \to \Sigma$ given by $\sigma \mapsto q(\sigma)$ is bijective.

Then $M := X(\Delta)/H$, where $X(\Delta)$ is a toric variety corresponding to Δ and $H = \exp(\mathfrak{h})$ is a subgroup of algebraic torus $G^{\mathbb{C}}$. Well known examples of manifolds with maximal torus actions are *moment-angle manifolds* and *LVMB-manifolds* which correspond to the cases when Δ is a subfan of the fan defining toric variety $\mathbb{C}^m$ or $\mathbb{C}P^m$, respectively.

Consider an action of a group $H' := \exp(p(\mathfrak{h})) \subset G$ on M. It always defines a holomorphic foliation $\mathcal{F}_{\mathfrak{h}}$. We refer to it as the *canonical foliation* on M.

Canonical foliation $(M, \mathcal{F}_{\mathfrak{h}})$ (or its leaf space) is considered as *nonrational toric manifold*. This is justified since if Σ is a rational fan $M/H' \cong V_\Sigma$ according to a famous result of Audin [2]. From this perspective it becomes essential to generalize well known properties of toric manifolds to the nonrational case. The cenrtal result of this work is the following theorem on de Rham cohomology of nonrational toric manifolds (more specifically, basic de Rham cohomology of foliation $\mathcal{F}_{\mathfrak{h}}$) which extends the result by Danilov [4] and Jurkiewicz [5]. It confirms a conjecture raised in [3].

Theorem 1. *Let M be a complex manifold with a maximal torus action, and let $\{a_1, \dots, a_m\}$ be the set of generators of Σ. There is an isomorphism of algebras:*

$$H^*_{\mathcal{F}_{\mathfrak{h}}}(M) \cong \mathbb{R}[v_1, ..., v_m]/(I_{\mathcal{K}} + J),$$

where $I_{\mathcal{K}}$ is the Stanley–Reisner ideal of the underlying simplicial complex of Σ, and J is the ideal generated by the linear forms

$$\sum_{i=1}^{m} \langle \boldsymbol{u}, q(\boldsymbol{e}_i)\rangle v_i, \quad \boldsymbol{u} \in (\mathfrak{g}/p(\mathfrak{h}))^* = \mathfrak{g}'^*.$$

Cartan model. Consider an action of some Lie algebra $\mathfrak{p}$ on a differential graded algebra A. Then according to ... its equivariant cohomology $H^*_{\mathfrak{p}}(A)$ may be computed as a cohomology of a *Cartan model*

$$\mathcal{C}_{\mathfrak{p}}(A) = ((S(\mathfrak{p}^*) \otimes A)^{\mathfrak{p}}, d_{\mathfrak{p}}),$$

where $(S(\mathfrak{p}^*)\otimes A)^{\mathfrak{p}}$ denotes the $\mathfrak{p}$-invariant subalgebra. Considering an element $\omega\in\mathcal{C}_{\mathfrak{p}}(A)$ as a $\mathfrak{p}$-equivariant polynomial map from $\mathfrak{g}$ to A, the differential $d_{\mathfrak{p}}$ is given by

$$d_{\mathfrak{p}}(\omega)(\xi)=d_A(\omega(\xi))-\iota_\xi(\omega(\xi)).$$

Moment-angle manifolds. First, we show this result when M is a moment-angle manifold $\mathcal{Z}_{\mathcal{K}}$.

Lemma 2. *Consider the algebra*

$$\mathcal{N}:=\mathcal{C}_R\big(\Omega(\mathcal{Z}_{\mathcal{K}})^{T^m}\big)=\Omega(\mathcal{Z}_{\mathcal{K}})^{T^m}\otimes S^*(p(\mathfrak{h})^*).$$

There is an isomorphism

$$H^*_{\mathcal{F}_{\mathfrak{h}}}(\mathcal{Z}_{\mathcal{K}})\cong H(\mathcal{N}).$$

Basing on formality of $(\mathbb{C}P^\infty)^{\mathcal{K}}$ shown in [7] the following lemma may be proved

Lemma 3. *The algebra $\mathcal{C}_{T^m}(\Omega(\mathcal{Z}_{\mathcal{K}}))$ is formal. Moreover, there is a zigzag of quasi-isomorphisms of DGAs between $\mathcal{C}_{T^m}(\Omega(\mathcal{Z}_{\mathcal{K}}))$ and $H_{T^m}(\mathcal{Z}_{\mathcal{K}})$ which respect the $S(\mathfrak{t}^*)$-module structure.*

Consider the following diagram

$$\begin{array}{ccc}\mathcal{N}\cong\mathbb{R}\otimes_{S^*(\mathfrak{g}'^*)}\mathcal{C}_{T^m}(\Omega(\mathcal{Z}_{\mathcal{K}})) & \longleftarrow & \mathcal{C}_{T^m}(\Omega(\mathcal{Z}_{\mathcal{K}}))\cong S(\mathfrak{g}'^*)\otimes\mathcal{N}\\ \uparrow & & \uparrow\\ \mathbb{R} & \longleftarrow & S(\mathfrak{g}'^*)\end{array}$$

Applying all the above lemmata and the famous corollary of Eilenberg-Moore spectral sequence on Tor-algebras it can be shown that there exists an analogous diagram at the level of cohomology of these algebras. Namely, we prove that there are following isomorphisms

$$\mathrm{Tor}_{S(\mathfrak{g}'^*)}\big(\mathbb{R},H^*_{T^m}(\mathcal{Z}_{\mathcal{K}})\big)\cong\mathrm{Tor}_{S(\mathfrak{g}'^*)}\big(\mathbb{R},S(\mathfrak{g}'^*)\otimes\mathcal{N}\big)\cong H(\mathcal{N}).$$

□

General case. Let $(M_1,\mathcal{F}_1)$ and $(M_2,\mathcal{F}_2)$ be smooth manifolds with foliations $\mathcal{F}_1$ on M_1 and $\mathcal{F}_2$ on M_2. We say that $(M_1,\mathcal{F}_1)$ and $(M_2,\mathcal{F}_2)$ are *transversely equivalent* if there exist a foliated manifold $(M_0,\mathcal{F}_0)$ and surjective submersions $f_i\colon M_0\to M_i$ for $i=1,2$ such that

- $f_i^{-1}(x_i)$ is connected for all $x_i\in M_i$, and
- the preimage under f_i of every leaf of $\mathcal{F}_i$ is a leaf of $\mathcal{F}_0$

Lemma 4. *If foliated manifolds $(M_1,\mathcal{F}_1)$, $(M_2,\mathcal{F}_2)$ are transversely equivalent via $(M_0,\mathcal{F}_0)$ and maps $f_i\colon M_0\to M_i$, then there is a DGA isomorphism $\Omega^*_{\mathcal{F}_1}(M_1)\cong\Omega^*_{\mathcal{F}_2}(M_2)$.*

Lemma 5. *For any manifold with maximal torus action M, there exists a transversely equivalent moment-angle manifold with the same $\{\Sigma;a_1,\dots,a_m\}$-data.*

Applying these two lemmata and the statement of the theorem for the moment-angle case we get the desired result. □

This work is based on a joint paper with Hiroaki Ishida and Taras Panov [6].

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