p-adic cohomology theories of stacks

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Reminder of stacks

Definition

A *stack* $\mathcal{X}$ is a functor from the category CAlg of commutative rings to the category of (∞-)groupoids satisfying sheaf condition with respect to some Grothendieck topology on CAlg.

To any stack $\mathcal{X}$ one can functorially attach a category $\mathrm{QCoh}(\mathcal{X})$ of *quasi-coherent sheaves on* $\mathcal{X}$. If $\mathcal{X}$ is the usual scheme, then $\mathrm{QCoh}(\mathcal{X})$ is equivalent to a usual unbounded derived category of quasi-coherent sheaves.

Reminder of stacks

Examples

Example

Let G be an algebraic group. *Algebraic classifying stack* BG is functor sending R to the groupoid of G-torsors on $\operatorname{Spec} R$. There is a canonical equivalence $\mathrm{QCoh}(BG) \simeq \mathrm{Rep}_G$.

Example

More generally, for a scheme X with an algebraic group G action there is a *quotient stack* $[X/G]$ such that $\mathrm{QCoh}([X/G])$ is a category of G-equivariant quasi-coherent sheaves on X.

Example

For a scheme X and an algebraic group G there is a stack $\mathrm{Bun}_G(X)$ of *G-bundles on X* defined as $\mathrm{Map}(X, BG)$.

Motivation: Totaro's conjecture

In [Tot18] Burt Totaro computed de Rham cohomology of various classifying stacks, which led him to the following conjecture:

Conjecture

Let G be a reductive group over the ring of p-adic integers $\mathbb{Z}_p$, then

$$\dim_{\mathbb{F}_p} H^i(BG(\mathbb{C}), \mathbb{F}_p) \leq \dim_{\mathbb{F}_p} H^i_{\mathrm{dR}}(BG_{\mathbb{F}_p}).$$

p-adic Hodge theory for schemes

Let $\mathfrak{S}$ denotes the ring $\mathbb{Z}_p[[u]]$.

Theorem 1 ([BMS16], [BS19])

Let X be a smooth proper scheme over $\mathbb{Z}_p$. There exists a perfect complex of $\mathfrak{S}$-modules $R\Gamma_{\Delta}(X/\mathfrak{S})$ such that

1. *(de Rham comparison)*

$$R\Gamma_{\Delta}(X/\mathfrak{S}) \otimes_{\mathfrak{S}} \mathbb{Z}_p \simeq R\Gamma_{\mathrm{dR}}(X/\mathbb{Z}_p).$$

2. *(etale comparison)*

$$R\Gamma_{\Delta}(X/\mathfrak{S}) \otimes_{\mathfrak{S}} W(\mathbb{C}_p^{\flat}) \simeq R\Gamma_{et}(X_{\mathbb{C}_p}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(\mathbb{C}_p^{\flat}).$$

Corollary

For X as in the theorem above one has

$$\dim_{\mathbb{F}_p} H^i_{et}(X_{\mathbb{C}_p}, \mathbb{F}_p) \leq \dim_{\mathbb{F}_p} H^i_{\mathrm{dR}}(X_{\mathbb{F}_p}/\mathbb{F}_p).$$

p-adic Hodge theory for stacks

Goal: generalize Theorem 1 to the setting of stacks.

Definition

A smooth stack $\mathcal{X}$ over a Noetherian base ring R is called *Hodge-proper* if for all i, j the Hodge cohomology module $H^{i,j}(\mathcal{X}) := H^j(\mathcal{X}, \wedge^i \mathbb{L}_{\mathcal{X}/R})$ is finitely generated over R.

Theorem 2

Let $\mathcal{X}$ be a smooth Hodge-proper stack over $\mathbb{Z}_p$. Then there exists a perfect complex of $\mathfrak{S}$-modules $R\Gamma_{\Delta}(\mathcal{X}/\mathfrak{S})$ such that de Rham comparison holds. Moreover, if $\mathcal{X}$ is a quotient stack of smooth proper scheme by a reductive group action, then the étale comparison holds as well.

Applications

Main

Theorem 3 (Generalized Totaro's conjecture)

Let X be a smooth proper scheme over $\mathbb{Z}_p$ with a reductive group G action. Then

$$\dim_{\mathbb{F}_p} H^i_{G(\mathbb{C})}(X, \mathbb{F}_p) \leq \dim_{\mathbb{F}_p} H^i_{\mathrm{dR}}([X/G]).$$

Theorem 4 (Equivariant Hodge degeneration)

Let X be a smooth proper scheme over $\mathbb{C}$ with a reductive group G action, such that X is proper over $\Gamma(X, \mathcal{O}_X)$ and $\Gamma(X, \mathcal{O}_X)^G$ is finite dimensional. Then there is a (non-canonical) splitting

$$H^n_{G(\mathbb{C})}(X, \mathbb{C}) \simeq \bigoplus_{p+q=n} H^{p,q}([X/G]).$$

Applications

Minor

- There is a well behaved theory of prismatic characteristic classes $c_\bullet^{\Delta}$.
- A theory of local systems and constructible sheaves on stacks.
- Extension of the standard spreading out results for schemes from SGA IV to the setting of stacks.

Further directions

- A non-trivial part of Theorem 2 is the étale comparison. For now we prove it by a series of reductions to the case BG, for G being a split torus. One can probably use recent advances in algebraic K-theory to give a more elegant proof. In particular it should be possible in Theorem 3 to weaken properness assumption on X.
- One can probably extend methods of the proof of Theorem 4 to show Hodge degeneration for the space of leaf of nice enough foliations. For this end one first need to develop a theory of Lie algebroids in positive characteristic.
- Apply these methods to the stack $\mathrm{Bun}_G(X)$ of G-bundles.

References

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Thank you!