CY structures on categories and geometric structures on moduli spaces

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Motivation

CY category A nc[1] analogue of an oriented space (topological, symplectic, holomorphic/CY), incarnated as a dg category $\mathcal{C}$ with a 'nc volume form' θ. (dg category means have Hom-complex $Hom^*(x, y)$ between objects $x, y \in \mathcal{C}$.)

[1] nc=non-commutative

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Moduli space of objects Any dg category $\mathcal{C}$ has a 'moduli space of objects' $\mathcal{M}_{\mathcal{C}}$. A nc volume form θ on $\mathcal{C}$ gives closed differential p-forms on $\mathcal{M}_{\mathcal{C}}$, for all p.

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Shifted symplectic form For $p = 2$, a shifted symplectic hence shifted Poisson structure. Useful for Donaldson-Thomas theory and quantization.

Geometric volume form For $p = \dim(\mathcal{C})$, honest volume form on the moduli space of point-like/toroidal objects in $\mathcal{C}$. From CY categories, get CY manifolds. Construction of mirrors. To be explored further.

Relative Calabi-Yau structures

Relative CY structures $\mathcal{C}$ smooth dg category with non-degenerate $\theta \in HC_*^-(\mathcal{C})[-d]$.[2]

[2] $HC_*^-(\mathcal{C})$: negative cyclic homology. A nc analogue of closed differential forms.

Relative Calabi-Yau structures

Relative CY structures A nc CY is a smooth dg category $\mathcal{C}$, non-degenerate $\theta \in HC_*^-(\mathcal{C})[-d]$.[3] Relative CY: $\mathcal{C} \to \mathcal{D}$ with non-degenerate $\lambda \in HC_*^-(\mathcal{D}, \mathcal{C})[-d-1]$. A nc analogue of oriented manifold with boundary.

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Examples Local systems on $\partial M \subset M$, an oriented manifold with boundary. Coherent sheaves on $D \subset X$, an anti-canonical divisor in a Fano variety.

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Relative Calabi-Yau structures: main theorem

Theorem (Brav-Dyckerhoff, Compositio, February 2019)

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Can glue oriented nc manifolds along boundary to produce closed oriented nc manifolds: a kind of nc cobordism.

Relative Calabi-Yau structures II

Moduli space $\mathcal{M}_{\mathcal{C}}$ parametrises 'compactly supported objects' of $\mathcal{C}$. Tangent complex at $x \in \mathcal{C}$ is

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Poincaré-Serre pairing The nc volume form θ and Poincaré-Serre duality induce a non-degenerate, skew symmetric pairing

$$T_x(\mathcal{M}_{\mathcal{C}})^{\otimes 2} \simeq Hom^*(x, x)[1]^{\otimes 2} \xrightarrow{\circ} Hom^*(x, x)[2] \xrightarrow{\theta \cap} k[d-2]$$

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Global symplectic forms For $d = 2$, this is just a skew-symmetric non-degenerate pairing (no shift). Goldman (local systems on surfaces) and Mukai (coherent sheaves on K3 surfaces) showed the above pairings on a part of the moduli space come from a global symplectic form.

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Higher dimension Pantev-Toën-Vaquié-Vezzosi developed the notion of shifted symplectic structure and shifted Lagrangian in derived algebraic geometry. Constructed higher dimensional geometric examples. Should be nc analogue.

Relative Calabi-Yau structures II: main theorem

Theorem (Brav-Dyckerhoff, arxiv:1812.11913)

Given a Calabi-Yau category $(\mathcal{C}, \theta)$ *of dimension* d*, the moduli space* $\mathcal{M}_{\mathcal{C}}$ *has an induced symplectic form of degree* $2 - d$*.*

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Given a relative Calabi-Yau structure $\lambda \in HC_*^-(\mathcal{D}, \mathcal{C})[-d-1]$ *on* $\mathcal{C} \to \mathcal{D}$*, with* $\mathcal{C}$ *CY of dimension* d*,* $\mathcal{M}_{\mathcal{D}} \to \mathcal{M}_{\mathcal{C}}$ *is shifted Lagrangian.*

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Examples $\mathcal{M}_{Loc(M)} \to \mathcal{M}_{Loc(\partial M)}$ (oriented manifold with boundary) and $\mathcal{M}_{Coh(X)} \to \mathcal{M}_{Coh(D)}$ (anti-canonical divisor in Fano variety) are Lagrangian.

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Application Can define relative Calabi-Yau structures on functors from generalised Kronecker quiver representations to coherent sheaves on elliptic curves. Get generalised Feigin-Odesski Poisson structures on moduli of Kronecker quiver representations, as well as formal symplectic groupoid integration thereof.

CY deformations, nc calculus, cyclic Deligne conjecture

Three papers with Nick Rozenblyum, currently being edited.

CY deformations Lie algebra structure on $HC_*(\mathcal{C})[2-d]$ controls exact CY defs of $(\mathcal{C}, \theta)$. Generalises Goldman-Chas-Sullivan string bracket to chain level and nc spaces. Natural trace-of-action map $HC_*(\mathcal{C})[2-d] \to \Gamma(\mathcal{M}_{\mathcal{C}}, \mathcal{O}_{\mathcal{M}_{\mathcal{C}}})[2-d]$ intertwines string bracket with shifted Poisson bracket.

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2) nc Gauss-Manin Construct nc Gauss-Manin connection with Griffiths transversality on periodic cyclic homology of universal infinitesimal deformation of $\mathcal{C}$. Map this compatibly to Gauss-Manin connection on de Rham cohomology of universal infinitesimal deformation of $\mathcal{M}_{\mathcal{C}}$.

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3) cyclic Deligne conjecture Establish the cyclic Deligne conjecture for smooth CY categories $(\mathcal{C}, \theta)$: Hochschild cochains $HH^*(\mathcal{C})$ is a circle equivariant E_2-algebra. Also a relative cyclic Deligne conjecture for relative CY structures. Gives 'string topology' operations on $HC_*(\mathcal{C})$ inducing the Lie algebra structure on $HC_*(\mathcal{C})[2-d]$.

Work in progress (with N. Rozenblyum)

Shifted Poisson structures CY structures on dg categories $\mathcal{C}$ induce shifted symplectic structures on moduli spaces $\mathcal{M}_{\mathcal{C}}$. We show 'log CY structures' induce shifted Poisson structures on $\mathcal{M}_{\mathcal{C}}$. Exotic examples to be further explored: moduli spaces of principal bundles on higher genus curves carry Poisson structures of degree -1.

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nc string topology 'Classical' string topology operations for closed oriented manifolds. Costello's approach to GW potential of proper CY category. Seidel's paper on formal groups and quantum Steenrod operations. Relative cyclic Deligne gives all of these structures at the chain level for nc oriented manifolds with boundary. Examples to be computed: mod p 'quantum Steenrod' operations on Hochschild chains $HH_*(X)$ of a Fano variety X, depending on an anti-canonical divisor $D \subset X$.