Some Exercises

Voronovo, 2019

Curves are usually smooth projective curves.

1) Let C be the curve over $\mathbb{F}_p$ defined by $x^3 + y^3 + z^3 = 0$. Show that $\#C(\mathbb{F}_p) = p + 1$ if $p \not\equiv 1 \pmod 3$.

2) Let $C = \mathbb{P}^1$ over $\mathbb{F}_q$. Show that points of degree $d > 1$ correspond bijectively to monic irreducible polynomials of degree d in $\mathbb{F}_q[x]$.

3) Let C be the curve defined by $y^3 + y = x^4$ over $\mathbb{F}_3$. Show that $t = x/y$ is a local parameter at the point at ∞. Show that dx defines a regular differential form on C. Calculate the genus g of C. Give a basis of the space of regular differential forms.

4)

i) Let C be the curve defined over k with $\mathrm{char}(k) \neq 2$ by $y^2 = f$ such that f is of degree 3 in $k[x]$ and has non-zero discriminant. Calculate the genus of C.

ii) Let C be the curve defined by $y^2 + y = (x^3 + x^2 + 1)/(x^3 + x + 1)$ over $\mathbb{F}_2$. Calculate $\#C(\mathbb{F}_2)$ and $\#C(\mathbb{F}_4)$.

5) Let F be a function field in one variable, R a valuation ring of F and m its maximal ideal. Let $0 \neq x \in m$.

i) Suppose that $x_1 = x, x_2, x_3, \ldots, x_n \in R$ are such that $x_i \in x_{i+1}m$. Prove that $n \leq [F : k(x)]$.

ii) Prove that m is a principal ideal.

6) Let C be the curve over $\mathbb{F}_5$ given by $y^5 + y = x^3$. Calculate the genus of C and a basis of the space of regular differentials.

7) Let $f : X \to Y$ be a non-constant morphism of smooth projective curves. Let P be a place of Y and Q be a place of X lying over it. Let t be a local parameter at P and s one at Q. Define

i) $e_Q = \mathrm{ord}_Q(f^*t)$; *(ramification index)*

ii) $r_Q = \mathrm{ord}_Q(f^*dt/ds)$.

Define $R := \sum_Q r_Q\, Q$ with the sum over all places Q of X.

i) Show that $r_Q = e_Q - 1$ if $\mathrm{char}(k) = p$ does not divide e_Q, else $r_Q > e_q - 1$.

ii) Show that $2g(X) - 2 = \deg(f)(2g(Y) - 2) + \deg(R)$. (*Hurwitz-Zeuthen formula*)

8) Let C be the curve over $\mathbb{F}_p$ given by $y^p - y = x^{p+1}$ and let $f : C \to \mathbb{P}^1$ be given by x (or by $\mathbb{F}_p(x) \subset \mathbb{F}_2(C)$). Calculate the degree of f, and the ramification index at ∞. Calculate the genus of C.

9) Let D be a divisor on the smooth projective curve C over the field k. Show that $\ell(D) = \dim_k L(D)$ depends only on the linear equivalence class of D.

10) Let C be a curve over $\mathbb{F}_q$ and P a place of C of degree d over $\mathbb{F}_q$. Furthermore, let Q be a place on $C/\mathbb{F}_{q^n}$ lying over P.

i) Show that $d' := \deg Q$ is equal to $d/\gcd(d,n)$.

ii) Show that there are $\gcd(d,n)$ places Q lying over P.

11) Let C/F_q and $N_n = \#C(\mathbb{F}_{q^n})$ and π_n the number of places of degree n on $C/\mathbb{F}_q$. Show that

i) $N_n = \sum_{d|n} d\,\pi_d$;

ii) $\pi_n = \frac{1}{n}\sum_{d|n} \mu(n/d)\, N_d$ with μ the Möbius function.

12) Show for $d, n \in \mathbb{Z}_{\geq 1}$ the following identity in $\mathbb{Z}[X]$:

$$(X^{n/\gcd(d,n)} - 1)^{\gcd(d,n)} = \prod_{\zeta:\zeta^n=1} (X - \zeta^d)\,.$$

13)

i) Let P be a point of degree n on $\mathbb{P}^1$ over $\mathbb{F}_q$. Show that $\ell(P) = n+1$.

ii) Prove for a divisor D on a curve C/k:

$$\dim_k L(D) \leq \deg(D) + 1\,.$$

14) Let D_1 and D_2 be effective divisors on a smooth projective curve C.

i) Show that $\dim|D_1| + \dim|D_2| \leq \dim|D_1 + D_2|$;

ii) If D is effective and $\ell(K-D) > 0$ then show that $\ell(D) + \ell(K-D) \leq g+1$.

iii) If If D is effective and $\ell(K-D) > 0$ then show the inequality $\ell(D) \leq 1 + \deg(D)/2$ (*Clifford*).

14) Let $Z(C,t)$ be the zeta function of $C/\mathbb{F}_q$. Show that

$$N_r = \frac{1}{(r-1)!}\frac{d^r}{dt^r}\log Z(C,t)_{|t=0}\,.$$

15) Calculate $Z(C,t)$ for C over $\mathbb{F}_2$ given by $y^2 + y = x^3$.

16) Let C be the curve in $\mathbb{P}^2$ over $\mathbb{F}_2$ given by $x^3 + y^3 + z^3 = 0$. Find a closed formula for $\#C(\mathbb{F}_{2^n})$.

17) Calculate $Z(C,t)$ for C/F_2 given by $y^2 + y = x^5$. Find a closed formula for $\#C(\mathbb{F}_{2^n})$. Calculate $\#\mathrm{Pic}(C)(\mathbb{F}_{2^n})$.

18) (2-*variable zeta function*) Define for $C/\mathbb{F}_q$ the function

$$Z(C,t,u) = \sum_{[D]} \frac{u^{\ell(D)} - 1}{u-1} t^{\deg(D)}$$

where the sum is over $[D] \in \mathrm{Pic}(C)(\mathbb{F}_q)$.

i) Show that $Z(C,t,q) = Z(C,t)$.
ii) Show that

$$(u-1)Z(C,t,u) = \sum_{i=0}^{2g-2} \sum_{[D],\deg(D)=i} u^{\ell(D)} t^{\deg(D)} + \sum_{i>2g-2} h u^{i+1-g} t^i - \sum_{i\geq 0} h t^i \,.$$

iii) Define

$$W_C(x,y) := \sum_{[D]} x^{\ell(D)} y^{\ell(K-D)} \,.$$

Prove:

$$W_C(x,y) = \sum_{i=0}^{2g-2} x^{\ell(D)} y^{\ell(K-D)} + h\frac{x^g}{1-x} + h\frac{y^g}{1-y} \,.$$

iv) Show that $(u-1)t^{1-g} Z(C,t,u) = W(ut, t^{-1})$.
v) Let E be an elliptic curve over $\mathbb{F}_q$. Show that

$$Z(E,t,u) = \frac{1+(h-1-u)\,t + u\,t^2}{(1-t)(1-ut)}$$

vi) Show the functional equation

$$Z(C,t,u) = u^{g-1} t^{2g-2} Z(C, 1/ut, u) \,.$$

19) Let C be a hyperelliptic curve $C \xrightarrow{2:1} \mathbb{P}^1$ over k with $\mathrm{char}(k) \neq 2$. Let P be a ramification point of $C \to \mathbb{P}^1$. Show that the gap sequence at P is $\{1,3,5,\ldots,2g-1\}$.

20) Let D be a divisor of degree d and $\dim|D| = n$. Show the following facts.
i) $n = d-g$ for $d > 2g-2$;
ii) $\dim \mathcal{L}_i = d-g-i$ for $d \geq 2g+i-1$;
iii) $j_i = i$ for $d \geq 2g+i$.

21) Let $D_t^{(n)}$ be the nth Hasse derivative. Show that

$$D_t^{(n)}(t^{-i}) = (-1)^n \binom{n+i-1}{n} t^{-n-i} \,.$$

22) Let F be a homogeneous polynomial in x_0, x_1, x_2 of degree d defining a curve C over k of characteristic $p > 0$. Let H be the Hessian

$$\det\left(\partial^2 F/\partial x_i \partial x_j\right) \,.$$

Write $F_i = \partial F/\partial x_i$ etc. Use Euler's formula to show that

$$X_0\, H = (d-1)\det\begin{pmatrix} F_0 & F_{01} & F_{02} \\ F_1 & F_{11} & F_{12} \\ F_2 & F_{12} & F_{22} \end{pmatrix}$$

Conclude that H vanishes identically if $d \equiv 1 (\mathrm{mod}\, p)$.

23) Let F be a homogeneous polynomial in x_0, x_1, x_2 of degree d defining a curve C over k of characteristic $p > 0$. Let P be a non-singular point of C. Show that P is an inflection point if $H(P) = 0$ if $p \neq 2$ and d is odd.

24) Let p be a prime. Let $n, m \in \mathbb{Z}_{\geq 1}$ with p-adic expansion $n = \sum \nu_i p^i$ and $m = \sum \mu_i p^i$ with $0 \leq \nu_i, \mu_i \leq p-1$. Show that $\binom{n}{m} \not\equiv 0 \, (\mathrm{mod}\, p)$ if and only if $\nu_i \geq \mu_i$.

25) Let p be a prime. Show that for $q = p^m$ we have $\binom{nq}{q} \equiv n \, (\mathrm{mod}\, p)$.

26) Let C be the curve given by $y^5 + y = x^3$ over $\mathbb{F}_5$. Show that $1, x, y, y^2$ generate $L(K)$. Calculate the order sequence $\{\epsilon_0, \ldots, \epsilon_3\}$ for $\mathcal{L} = |K|$. Is $|K|$ classical? Is $|K|$ Frobenius classical?

27) Show that if the Frobenius number ν_i satisfies $\nu_i < p$ then $(\nu_0, \ldots, \nu_i) = (0, 1, \ldots, i)$.

28) Show that for the curve $y^5 + y = x^3$ over $\mathbb{F}_{25}$ the Frobenius sequence $\{\nu_0, \nu_1, \nu_2\}$ equals $\{0, 1, 5\}$.

29) Let C be a smooth plane curve of degree d over $\mathbb{F}_q$. Prove that

$$N_1 \leq \frac{1}{2}(\nu_1(2g-2) + n(q+2)) \,.$$

30) Let C be the hermitian curve over $\mathbb{F}_{q^2}$ given by

$$x^{q+1} + y^{q+1} + z^{q+1} = 0 \,.$$

Show that $\{\epsilon_0, \epsilon_1, \epsilon_2\} = \{0, 1, q\}$. Show that at a rational point P we have $\{j_0, j_1, j_2\} = \{0, 1, q+1\}$ and $\{\nu_0, \nu_1\} = \{0, q\}$. Calculate the Stöhr-Voloch bound for $\#C(\mathbb{F}_{q^2})$.

31) Calculate the automorphism group of the hermitian curve over $\mathbb{F}_{q^2}$.

32) Let $\mathcal{L} = |D|$ be a complete linear series on a curve C with $\deg(D) \geq 2g$.
  i) Show that such a system exists.
  ii) Show that $n = \dim |D| = d - g$ and $\mathcal{L}$ is base-point-free.

33) Determine for $p = 13$ for which genera g the Hasse-Weil-Serre bound is better than the Ihara bound.

34) Calculate the Hasse-Weil-Serre and Ihara bound for $(q, g) = (2, 3)$. Show that the curve over $\mathbb{F}_2$ given by

$$y^2z^2 + yz^3 + xy^3 + x^2y^2 + x^3z + xz^3 = 0$$

passes through all points of $\mathbb{P}^2(\mathbb{F}_2)$. Show that $N_2(3) = 7$.