Учебный центр НИУ ВШУ в Вороново

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Как заинтересовать студентов бирациональной геометрией, используя простую теорию чисел?

Кубические Формы

ПРЕДИСЛОВИЕ

I

Любой математик, неравнодушный к теории чисел, испытал на себе очарование теоремы Ферма о сумме двух натуральных квадратов. Психолог юнговской школы нашел бы, вероятно, что такие диофантовы задачи в высшей степени архитипичны.

Замысел предлагаемой книги возник из попытки разобраться, что происходит с суммами трех рациональных кубов. Излишне говорить, что результат далек от простоты, фундаментальности и завершенности классических образцов. Автор обобщал задачу всеми способами, которые приходили ему на ум, и применял все технические средства, какие только умел. Получившееся в итоге нагромождение неассоциативных законов композиции, моноидальных преобразований и когомологий Галуа составило эту книжку.

II

Задача о суммах трех кубов имеет почтенную историю. Вот основной результат, оставленный классиками (см. Диксон [1]).

Теорема. *Любое рациональное число является суммой трех кубов рациональных чисел.*

Первое доказательство (Райли, 1825; Ричмонд, 1930):

$$a=\left(\frac{a^3-3^6}{3^2a^7+3^4a+3^6}\right)^3+\left(\frac{-a^3+3^5a+3^6}{3^2a^2+3^4a+3^6}\right)^3+\left(\frac{a^2+3^4a}{3^2a^2+3^4a+3^6}\right)^3.$$

Юрий Иванович Манин

The Ladies' Diary, 1825

Теорема (С. Райли)

Every rational number is a sum of three cubes of rational numbers.

Доказательство.

Let q be a rational number. Let

$$\alpha = \frac{1}{36}\frac{512q^4 - 1600q^3 + 108440q^2 - 173691q - 729}{128q^3 - 416q^2 + 8082q - 243},$$

$$\beta = -\frac{q(64q^2 - 1648q - 7263)}{128q^3 - 416q^2 + 8082q - 243},$$

$$\gamma = -\frac{1}{36}\frac{512q^4 - 1600q^3 - 15976q^2 + 246213q - 729}{128q^3 - 416q^2 + 8082q - 243}.$$

Заметим, что $128q^3 - 416q^2 + 8082q - 243 \neq 0$. Почему?
Using Maple, one can check that

$$\alpha^3 + \beta^3 + \gamma^3 = q.$$

Пифагоровы тройки

Пример (Пифагор)

Let m, n, k be any integers. Then

$$\boxed{\Big(k(m^2-n^2)\Big)^2+\Big(2kmn\Big)^2=\Big(k(m^2+n^2)\Big)^2.}$$

This gives all integral solutions to

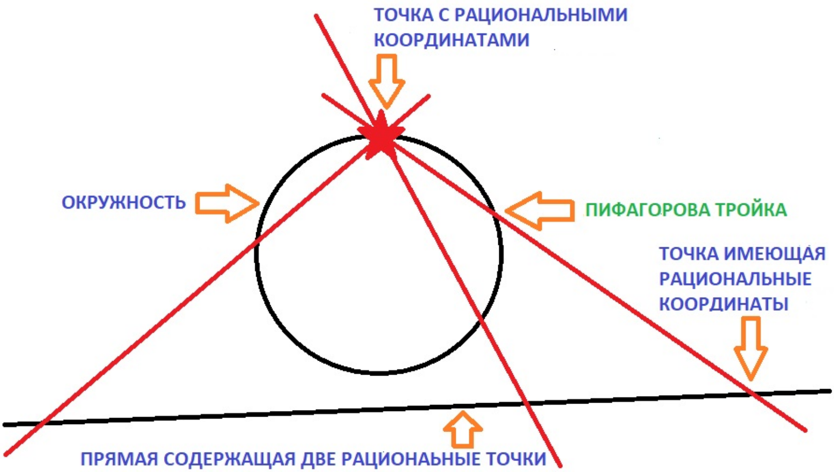
$$\boxed{x^2+y^2=z^2.}$$

- Let $\mathcal{C}$ be a circle in $\mathbb{R}^2$ given by $x^2+y^2=1$.
- All points in $\mathcal{C}\setminus(1,0)$ with rational coordinates are given by

$$\boxed{\left(\frac{t^2-1}{t^2+1},\frac{2t}{t^2+1}\right)}$$

for some $t\in\mathbb{Q}$.

Рациональная параметризация коник



Нерациональные кубики

Let $\mathcal{C}$ be the curve in $\mathbb{C}^2$ that is given by

$$\boxed{x^3 + y^3 = 1.}$$

Then $\mathcal{C}$ does not have rational parametrization.

Упражнение

Let $x(t)$, $y(t)$, $z(t)$ be coprime polynomials in $\mathbb{C}[t]$ such that

$$\boxed{x^3(t) + y^3(t) = z^3(t).}$$

Then $x(t)$, $y(t)$, $z(t)$ are constant.

Следствие

Let $x(t)$ and $y(t)$ be rational functions in $\mathbb{C}(t)$ such that

$$\boxed{x^3(t) + y^3(t) = 1.}$$

Then $x(t)$ and $y(t)$ are constant.

Рациональные кубики

Let $\mathcal{C}$ be the curve in $\mathbb{C}^2$ that is given by

$$\boxed{y^2 + xy^2 - y^3 - x^3 - x^2 = 0.}$$

Then $\mathcal{C}$ has rational parametrization.
Let L be the line $y = tx$. Then $L \cap \mathcal{C}$ is given by

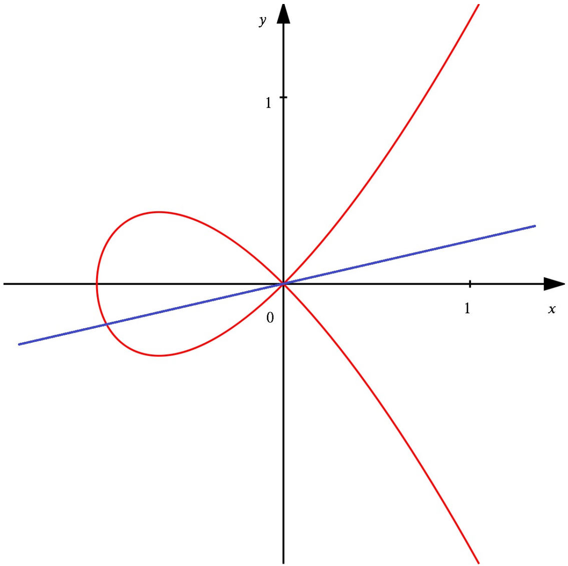
$$\begin{cases} y^2 + xy^2 - y^3 - x^3 - x^2 = 0, \\ y = tx. \end{cases}$$

Solving this system, we find the point

$$\boxed{\left(\frac{t^2 - 1}{t^3 - t^2 + 1}, \frac{t(t^2 - 1)}{t^3 - t^2 + 1}\right).}$$

When t runs through $\mathbb{C}$, we obtain all points in $\mathcal{C} \setminus (0, 0)$.

Рациональная параметризация нодальной кубики



Стереографическая проекция

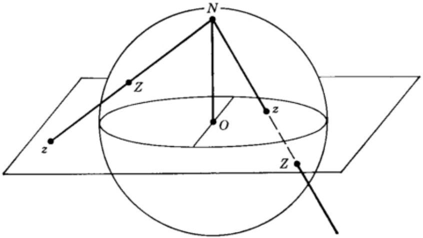
Let S_2 be the sphere in $\mathbb{C}^3$ that is given by

$$\boxed{x^2 + y^2 + z^2 = 1.}$$

Then S_2 has rational parametrization:

$$\boxed{\left(\frac{1 - u^2 - v^2}{1 + u^2 + v^2}, \frac{2u}{1 + u^2 + v^2}, \frac{2v}{1 + u^2 + v^2}\right).}$$

When (v, u) runs through $\mathbb{C}^2$, we obtain almost all points in S_2.



Рациональная параметризация квадрик

Let S_2 be the quadric surface in $\mathbb{P}^3_{\mathbb{C}}$ that is given by

$$\boxed{x^2 + y^2 + z^2 = t^2.}$$

Then S_2 has rational parametrization:

$$\Big[w^2 - u^2 - v^2 : 2uw : 2vw : w^2 + u^2 + v^2\Big].$$

When $[v : u : w]$ runs through $\mathbb{P}^2_{\mathbb{C}}$ without $w = 0$, we obtain

$$\boxed{S_2 \setminus \Big(L_1 \cup L_2\Big),}$$

where

- L_1 is the line $w = u + iv = 0$;
- L_2 is the line $w = u - iv = 0$.

Рациональные и унирациональные многообразия

Let X be an irreducible projective variety of dimension n.

Вопрос

Что такое рациональная параметризация многообразия X?

- A dominant rational map $\mathbb{P}^n_{\mathbb{C}} \dashrightarrow X$.
- A birational map $\mathbb{P}^2_{\mathbb{C}} \dashrightarrow X$.

Определение

- *X is rational $\iff$ $\exists$ birational map $\mathbb{P}^n_{\mathbb{C}} \dashrightarrow X$.*
- *X is unirational $\iff$ $\exists$ dominant rational map $\mathbb{P}^n_{\mathbb{C}} \dashrightarrow X$.*

Пример

Let S_2 be an irreducible quadric surface in $\mathbb{P}^3_{\mathbb{C}}$.
Then S_2 is rational.

Пример

Let $\mathcal{C}$ be an irreducible cubic curve in $\mathbb{P}^2_{\mathbb{C}}$.
Then $\mathcal{C}$ is rational $\iff$ $\mathcal{C}$ is singular.

Рациональность кубических поверхностей

Теорема

Let S_3 be a smooth cubic surface in $\mathbb{P}^3_{\mathbb{C}}$. Then S_3 is rational.

Доказательство.

Let L_1 and L_2 be two lines in S_3 such that

$$L_1 \cap L_2 = \varnothing.$$

Then $L_1 \times L_2$ is rational, since

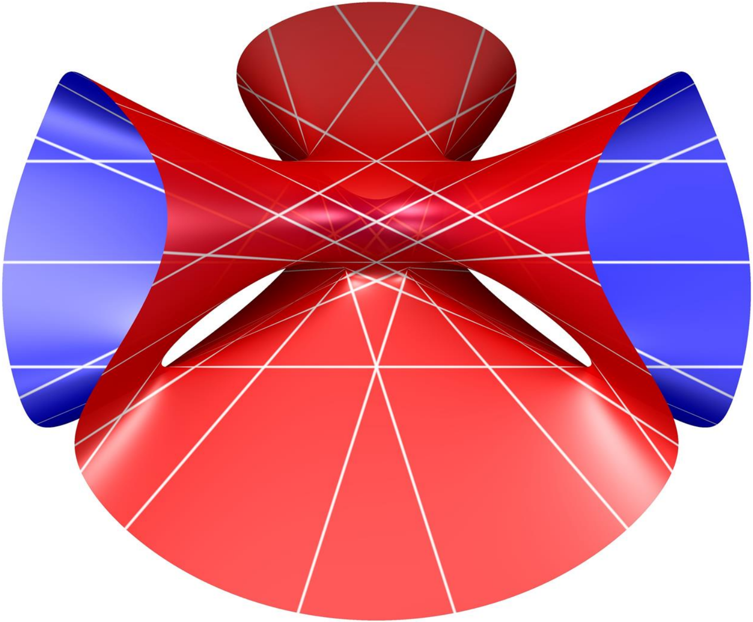
$$L_1 \cong L_2 \cong \mathbb{P}^1_{\mathbb{C}}.$$

Define a map $\psi\colon L_1 \times L_2 \dashrightarrow S_3$ as follows:

- Let (P, Q) be a general point in $L_1 \times L_2$.
- Let ℓ be the line in $\mathbb{P}^3_{\mathbb{C}}$ that contains P and Q.
- Let $\phi\big((P, Q)\big)$ be the third point in $\ell \cap S_3$.

Then $\phi\colon L_1 \times L_2 \dashrightarrow S_3$ is a birational map.
This shows that S_3 is rational. □

27 прямых на кубической поверхности



Рациональность поверхности $x^3 + y^3 + z + z^3 = 0$

Let S_3 be the surface in $\mathbb{C}^3$ given by $x^3 + y^3 + z + z^3 = 0$.

- Let L_1 be the line $x + y = z = 0$.
- Let L_2 be the line $\omega x + y = z - i = 0$, where $\omega = -\frac{1+i\sqrt{3}}{2}$.

Let $P = (a, -a, 0) \in L_1$ and $Q = (b, -\omega b, i) \in L_2$.
Let ℓ be the line in $\mathbb{C}^3$ that contains P and Q. Then ℓ is given by

$$\Big(a + \lambda(b - a), -a + \lambda(a - \omega b), \lambda i\Big),$$

where $\lambda \in \mathbb{C}$. Then $\ell \cap S_3$ consists of the points P, Q and

$$\left(\frac{(6\omega + 3)a^2b^2 + 2ia - ib}{(3\omega - 3)a^2b + (3\omega + 6)ab^2 + i}, \frac{(3\omega - 3)a^2b^2 + i\omega b - 2ia}{(3\omega - 3)a^2b + (3\omega + 6)ab^2 + i}, \frac{i(3\omega - 3)a^2b + 1}{(3\omega - 3)a^2b + (3\omega + 6)ab^2 + i}\right).$$

Рациональность поверхности $x^3 + y^3 + t^2z + z^3 = 0$

Let S_3 be the surface in $\mathbb{P}^3_{\mathbb{C}}$ given by $x^3 + y^3 + t^2z + z^3 = 0$.
There is a birational map $\mathbb{P}^2_{\mathbb{C}} \dashrightarrow S_3$ that maps $[a : b : c]$ to

$$\Big[(6\omega + 3)b^2c + 2iac^2 - ia^2b : (3\omega - 3)b^2c + i\omega a^2b - 2iac^2 :$$

$$: i(3\omega - 3)bc + a^2c : (3\omega - 3)bc^2 + (3\omega + 6)ab^2 + ia^2c\Big].$$

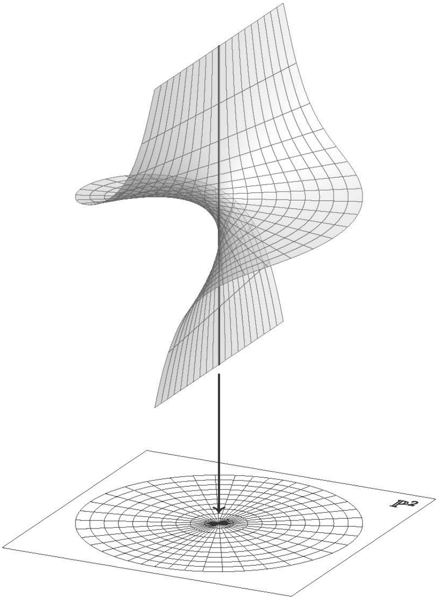
This map is undefined in the points

$$\begin{cases} (6\omega + 3)b^2c + 2iac^2 - ia^2b = 0, \\ (3\omega - 3)b^2c + i\omega a^2b - 2iac^2 = 0, \\ i(3\omega - 3)bc^2 + a^2c = 0, \\ (3\omega - 3)bc^2 + (3\omega + 6)ab^2 + ia^2c = 0. \end{cases}$$

This system of equations gives us are exactly 6 points in $\mathbb{P}^2_{\mathbb{C}}$.

- The inverse map $S_3 \dashrightarrow \mathbb{P}^2_{\mathbb{C}}$ is well defined.
- It contracts 6 disjoint lines in S_3 to the points above.

Раздутие точки на плоскости



Унирациональность кубических поверхностей

Let S_3 be a smooth cubic surface in $\mathbb{P}^3_{\mathbb{C}}$ that is defined over $\mathbb{Q}$.

Теорема (Бенжамино Сегре)

The surface S_3 is unirational over $\mathbb{Q}$ $\iff$ S_3 has a rational point.

Suppose that S_3 contains a rational point P.

- Let Π be the plane in $\mathbb{P}^3_{\mathbb{C}}$ that is tangent to S_3 in P.
- Put $C = S_3 \cap \Pi$. Then C is a singular cubic curve.
- Then C is defined over $\mathbb{Q}$, since P is defined over $\mathbb{Q}$.
- Suppose that C is irreducible. Then C is rational over $\mathbb{Q}$.
- This gives us a infinitely many rational points in S_3.
- Pick one of them $Q \neq P$ and repeat the construction.
- This gives singular cubic curve $Z \subset S_3$ defined over $\mathbb{Q}$.

Now we can construct a dominant rational map

$$C \times Z \dashrightarrow S_3$$

as in the proof of rationality of complex smooth cubic surfaces.

Бенжамино Сегре

Теорема Райли I

Let S_3 be the surface in $\mathbb{P}^3_{\mathbb{C}}$ given by

$$x^3 + y^3 + z^3 - qt^3 = 0,$$

where q is a non-zero rational number.
Then S_3 is unirational over $\mathbb{Q}$ by Segre's Theorem.
Let us show this. To do this, replace S_3 by its part $z \neq 0$.
Thus, we may assume that S_3 is given by

$$x^3 + y^3 + 1 - qt^3 = 0.$$

Let ℓ be the line given by

$$\Big(-1 + 2\lambda, \lambda, 0 \Big),$$

where $\lambda \in \mathbb{Q}$. Then $\ell \cap S_3 = (-1, 0, 0)$ over $\mathbb{Q}$.
Over $\mathbb{Q}(\sqrt{-2})$ the intersection $\ell \cap S_3$ contains two more points:

$$\left(\frac{1 \pm 2\sqrt{-2}}{3}, \frac{2 \pm \sqrt{-2}}{3}, 0 \right).$$

Теорема Райли II

Let $\widehat{x} = x - \frac{1+2\sqrt{-2}}{3}$, $\widehat{y} = y - \frac{2+\sqrt{-2}}{3}$, $\widehat{t} = t$. Then S_3 is given by

$$\Big(-\frac{7}{3} + \frac{4}{3}\sqrt{-2}\Big)\widehat{x} + \Big(\frac{2}{3} + \frac{4}{3}\sqrt{-2}\Big)\widehat{y} + \\ + \big(1+2\sqrt{-2}\big)\widehat{x}^2 + \big(2+\sqrt{-2}\big)\widehat{y}^2 + \widehat{y}^3 + \widehat{x}^3 - q\widehat{t}^3 = 0.$$

Let Π be the tangent plane to S_3 at P. Then Π is given by

$$\widehat{y} = \frac{7-4\sqrt{-2}}{4\sqrt{-2}+2}\widehat{x}.$$

Thus, the intersection $\Pi \cap S_3$ is given by

$$\Big(-10\sqrt{-2} - 31\Big)\widehat{x}^3 + \Big(36 - 18\sqrt{-2}\Big)\widehat{x}^2 + 8q\widehat{t}^3 = 0.$$

Intersecting this curve with the line $t = \lambda x$ in Π, we get the point

$$\left(\frac{2-18\sqrt{-2}}{31-8q\lambda^3+10\sqrt{-2}}, \frac{36\lambda-18\sqrt{-2}}{31-8q\lambda^3+10\sqrt{-2}}, \frac{-27\sqrt{-2}-54}{31-8q\lambda^3+10\sqrt{-2}}\right).$$

Теорема Райли III

We see that the surface S_3 contain the point

$$\left(\frac{2-18\sqrt{-2}}{31-8q\lambda^3+10\sqrt{-2}}, \frac{36\lambda-18\sqrt{-2}}{31-8q\lambda^3+10\sqrt{-2}}, \frac{-27\sqrt{-2}-54}{31-8q\lambda^3+10\sqrt{-2}}\right)$$

in coordinates $\widehat{x}=x-\frac{1+2\sqrt{-2}}{3}$, $\widehat{y}=y-\frac{2+\sqrt{-2}}{3}$, $\widehat{t}=t$.
Rewriting this point in coordinated x, y and t, we obtain the point

$$\left(-\frac{2\sqrt{-2}+1}{3}\cdot\frac{8q\lambda^3+20\sqrt{-2}-19}{31-8q\lambda^3+10\sqrt{-2}},\right.$$
$$\frac{2\sqrt{-2}+4}{3}\cdot\frac{-4q\lambda^3+5\sqrt{-2}-25}{31-8q\lambda^3+10\sqrt{-2}},$$
$$\left.\frac{\lambda(36-18\sqrt{-2})}{31-8q\lambda^3+10\sqrt{-2}}\right)$$

contained in S_3 for every $\lambda\in\mathbb{C}$.

- Main trick: let $\boxed{\lambda=a+b\sqrt{-2}}$ (Weil restriction).

Теорема Райли IV

Recall that S_3 is the surface in $\mathbb{Q}^3$ given by $x^3 + y^3 + 1 = qt^3$. Let

$$x_1 = \frac{1}{3}\frac{(2\sqrt{-2}+1)(-16\sqrt{-2}b^3q - 48ab^2q + 24\sqrt{-2}a^2bq + 8a^3q + 20\sqrt{-2} - 19)}{-16\sqrt{-2}b^3q - 48ab^2q + 24\sqrt{-2}a^2bq + 8a^3q - 10\sqrt{-2} - 31},$$

$$y_1 = \frac{2}{3}\frac{(\sqrt{-2}+2)(-8\sqrt{-2}b^3q - 24ab^2q + 12\sqrt{-2}a^2bq + 4a^3q - 5\sqrt{-2} + 25)}{-16\sqrt{-2}b^3q - 48ab^2q + 24\sqrt{-2}a^2bq + 8a^3q - 10\sqrt{-2} - 31},$$

$$t_1 = \frac{18(a + b\sqrt{-2})(\sqrt{-2} - 2)}{-16\sqrt{-2}b^3q - 48ab^2q + 24\sqrt{-2}a^2bq + 8a^3q - 10\sqrt{-2} - 31}.$$

Then $(x_1, y_1, t_1) \in S_3$ for every rational a and b.
The complex conjugate point $(\overline{x}_1, \overline{y}_1, \overline{t}_1)$ also lies in S_3. Let

$$x_2 = \frac{1}{3}\frac{(-2\sqrt{-2}+1)(16\sqrt{-2}b^3q - 48ab^2q - 24\sqrt{-2}a^2bq + 8a^3q - 20\sqrt{-2} - 19)}{16\sqrt{-2}b^3q - 48ab^2q - 24\sqrt{-2}a^2bq + 8a^3q + 10\sqrt{-2} - 31},$$

$$y_2 = \frac{2}{3}\frac{(-\sqrt{-2}+2)(8\sqrt{-2}b^3q - 24ab^2q - 12\sqrt{-2}a^2bq + 4a^3q + 5\sqrt{-2} + 25)}{16\sqrt{-2}b^3q - 48ab^2q - 24\sqrt{-2}a^2bq + 8a^3q + 10\sqrt{-2} - 31},$$

$$t_2 = \frac{18(a - b\sqrt{-2})(\sqrt{-2} - 2)}{16\sqrt{-2}b^3q - 48ab^2q - 24\sqrt{-2}a^2bq + 8a^3q + 10\sqrt{-2} - 31}.$$

Then $(x_2, y_2, t_2) = (\overline{x}_1, \overline{y}_1, \overline{t}_1)$ is contained in S_3.

Теорема Райли V

Let L be the line that contains (x_1, y_1, t_1) and (x_2, y_2, t_2). Then L is defined over $\mathbb{Q}$.
The intersection $L \cap S_3$ consists of (x_1, y_1, t_1), (x_2, y_2, t_2) and $\left(\frac{\theta_1}{\epsilon}, \frac{\theta_2}{\epsilon}, \frac{\theta_3}{\epsilon}\right)$, where

$$\theta_1 = -512a^{12}q^4 + 6144a^{10}b^2q^4 + 30720a^8b^4q^4 + 81920a^6b^6q^4 + 122880a^4b^8q^4 + 98304a^2b^1 0q^4 + \\ + 32768b^1 2q^4 - 1600a^9q^3 + 1920a^8bq^3 + 10240a^6b^3q^3 + 38400a^5b^4q^3 + 15360a^4b^5q^3 + 102400a^3b^6q^3 + \\ + 76800ab^8q^3 - 10240b^9q^3 + 108440a^6q^2 + 30048a^5bq^2 - 317760a^4b^2q^2 - 760192a^3b^3q^2 + 1192800a^2b^4q^2 + \\ + 120192ab^5q^2 - 496000b^6q^2 - 173691a^3q + 633582a^2bq - 729324ab^2q + 286200b^3q - 729.$$

$$\theta_2 = 2304a^9q^3 + 34560a^8bq^3 + 184320a^6b^3q^3 - 55296a^5b^4q^3 + 276480a^4b^5q^3 - \\ - 147456a^3b^6q^3 - 110592ab^8q^3 - 184320b^9q^3 - 59328a^6q^2 - 146880a^5bq^2 + 100224a^4b^2q^2 + 419328a^3b^3q^2 - \\ - 200448a^2b^4q^2 - 587520ab^5q^2 + 474624b^6q^2 - 261468a^3q + 801900a^2bq - 793152ab^2q + 252720b^3q.$$

$$\theta_3 = -4608a^{10}q^3 - 4608a^9bq^3 - 27648a^8b^2q^3 - 36864a^7b^3q^3 - 36864a^6b^4q^3 - \\ - 110592a^5b^5q^3 + 73728a^4b^6q^3 - 147456a^3b^7q^3 + 221184a^2b^8q^3 - 73728ab^9q^3 + 147456b^{10}q^3 + \\ + 14976a^7q^2 - 19584a^6bq^2 + 165888a^5b^2q^2 - 105984a^4b^3q^2 + 281088a^3b^4q^2 - 207360a^2b^5q^2 + 18432ab^6q^2 - \\ - 147456b^7q^2 - 290952a^4q + 255960a^3bq + 820368a^2b^2q - 1402272ab^3q + 616896b^4q + 8748a - 8748b.$$

$$\epsilon = 512a^{12}q^4 + 6144a^{10}b^2q^4 + 30720a^8b^4q^4 + 81920a^6b^6q^4 + 122880a^4b^8q^4 + \\ + 98304a^2b^1 0q^4 + 32768b^1 2q^4 - 1600a^9q^3 + 1920a^8bq^3 + 10240a^6b^3q^3 + 38400a^5b^4q^3 + 15360a^4b^5q^3 + \\ + 102400a^3b^6q^3 + 76800ab^8q^3 - 10240b^9q^3 - 15976a^6q^2 - 343200a^5bq^2 + 55488a^4b^2q^2 + 608384a^3b^3q^2 + \\ + 446304a^2b^4q^2 - 1372800ab^5q^2 + 499328b^6q^2 + 246213a^3q - 626130a^2bq + 530388ab^2q - 133704b^3q - 729.$$

Теорема Райли VI

For every rational a and b such that $\epsilon \neq 0$, we have

$$\left(\frac{\theta_1}{\epsilon}\right)^3 + \left(\frac{\theta_2}{\epsilon}\right)^3 + 1 = q\left(\frac{\theta_3}{\epsilon}\right)^3.$$

Thus, for every rational a and b such that $\theta_3 \neq 0$, we have

$$q = \left(\frac{\theta_1}{\theta_3}\right)^3 + \left(\frac{\theta_2}{\theta_3}\right)^3 + \left(\frac{\epsilon}{\theta_3}\right)^3.$$

For example, let $a = 1$ and $b = 0$. Then

$$\frac{\theta_1}{\theta_3} = \frac{1}{36}\frac{512q^4 - 1600q^3 + 108440q^2 - 173691q - 729}{128q^3 - 416q^2 + 8082q - 243},$$

$$\frac{\theta_2}{\theta_3} = -\frac{q(64q^2 - 1648q - 7263)}{128q^3 - 416q^2 + 8082q - 243},$$

$$\frac{\epsilon}{\theta_3} = -\frac{1}{36}\frac{512q^4 - 1600q^3 - 15976q^2 + 246213q - 729}{128q^3 - 416q^2 + 8082q - 243}.$$

Проблема Люрота

Вопрос

Are there unirational varieties of dimension n that are not rational?

Теорема (Lüroth, 1876)

Every one-dimensional unirational variety is rational.

Теорема (Castelnuovo)

Every two-dimensional complex unirational variety is rational.

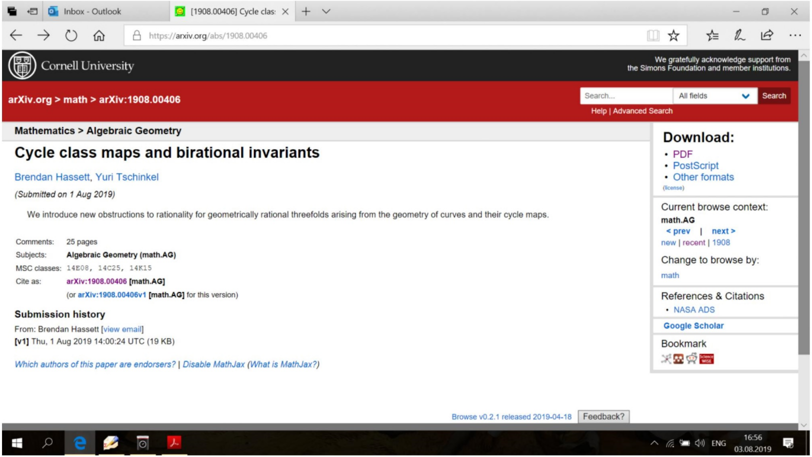
Пример

Let S_3 be a smooth cubic surface in $\mathbb{P}^3_{\mathbb{C}}$ defined by

$$t(x^2+y^2)=(z-\alpha_1 t)(z-\alpha_2 t)(z-\alpha_3 t),$$

where α_1, α_2 and α_3 are distinct real numbers.
Then $S_3(\mathbb{R})$ is not connected, so that S_3 is not rational over $\mathbb{R}$.
But S_3 is unirational over $\mathbb{R}$ by the proof of Segre's theorem.

arXiv, Friday 2nd August 2019



Нерациональные кубические поверхности

Let S_3 be a smooth cubic surface in $\mathbb{P}^3_{\mathbb{C}}$ that is defined over $\mathbb{Q}$.

Теорема (Бенжамино Сегре, Юрий Иванович Манин)

Suppose that for every curve $C \subset S_3$ defined over $\mathbb{Q}$ one has

$$C = S_3 \cap F$$

for some surface F in $\mathbb{P}^3_{\mathbb{C}}$. Then S_3 is not rational over $\mathbb{Q}$.

- This theorem holds for del Pezzo surfaces of degree $\leqslant 3$.

Пример

Let S_3 be the surface in $\mathbb{P}^3_{\mathbb{C}}$ that is given by

$$2x^3 + 3y^3 + 5z^3 + 7t^3 = 0.$$

Then S_3 is non-rational over $\mathbb{Q}$ by Segre–Manin theorem.
But S_3 is unirational, since $[1 : 1 : -1 : 0] \in S_3$.

Поверхности дель Пеццо степени 1 и 2

Let $\mathbb{F}$ be an arbitrary field.

Теорема (Manin)

Let S_4 be a smooth quartic surface in $\mathbb{P}(1,1,1,2)$ defined over $\mathbb{F}$. Suppose that S_4 contains $\mathbb{F}$-rational point in general position. Then S_4 is unirational over $\mathbb{F}$.

Теорема (Salgado, Testa, Varilly-Alvarado, Festi, van Luijk)

Let S_4 be a smooth quartic surface in $\mathbb{P}(1,1,1,2)$ defined over $\mathbb{F}$. Suppose that $\mathbb{F}$ is a finite field. Then S_4 is unirational over $\mathbb{F}$.

Вопрос

Let S_6 be a smooth sextic surface in $\mathbb{P}(1,1,2,3)$ defined over $\mathbb{F}$. Is S_6 unirational over $\mathbb{F}$?

- Let $\overline{\mathbb{F}}$ be the algebraic closure of the field $\mathbb{F}$.

Вопрос

Let S be a surface defined over $\mathbb{F}$ that is rational over $\overline{\mathbb{F}}$. Suppose that S contains $\mathbb{F}$-rational point. Is S unirational over $\mathbb{F}$?

Проблема Люрота в размерности три над $\mathbb{C}$

Теорема (Исковских & Манин, 1971)

Every smooth quartic hypersurface in $\mathbb{P}^4_{\mathbb{C}}$ is not rational.

Теорема (Clemens & Griffiths, 1972)

Every smooth cubic hypersurface in $\mathbb{P}^4_{\mathbb{C}}$ is not rational.

- Some smooth quartic hypersurface in $\mathbb{P}^4_{\mathbb{C}}$ are unirational.
- All smooth cubic hypersurface in $\mathbb{P}^4_{\mathbb{C}}$ are unirational.

Пример (Artin & Mumford, 1972)

Let $F_2 = x^2 + y^2 + z^2 + t^2 + (x + y + z + t)^2$ *and*

$$F_4 = x^4 + y^4 + z^4 + t^4 + \big(x + y + z + t\big)^4.$$

Let X *be the hypersurface in* $\mathbb{P}(1,1,1,1,2)$ *that is given by*

$$w^2 = F_4(x, y, z, t) - \frac{1}{2}F_2^2(x, y, z, t).$$

Then X *is unirational. But* X *is not rational.*

Унирациональность трехмерных многообразия Фано

Most of smooth Fano threefolds are known to be unirational over $\mathbb{C}$.

Вопрос

Let V_6 be a smooth hypersurface in $\mathbb{P}(1,1,1,2,3)$ of degree 6. Is V_6 unirational?

Вопрос

Let X_6 be a smooth hypersurface in $\mathbb{P}(1,1,1,1,3)$ of degree 6. Is X_6 unirational?

Вопрос

Let X_4 be a smooth hypersurface in $\mathbb{P}^4$ of degree 4. Is X_4 unirational?

- A variety X is said to be rationally connected if every two general points in X can be joined by a rational curve.

Вопрос

Let X be a rationally connected variety. Is X unirational?